22 – 26 April 2024, Trento, Italy

# ProHyp 2024

# Dead, follower, and pressure loads: surface waves and compressive vs. tensile bifurcations


Davide Bigoni (1), George Cristian Ivan (2), Roberta Springhetti (1)

Contact e-mail: davide.bigoni@unitn.it

(1) Instabilities Lab, University of Trento, via Mesiano 77, Trento, Italy

(2) Faculty of Mechanical, Industrial and Maritime Engineering, Ovidius University of Constanta, 900527 Constanta, Romania



## Abstract

Paralleling the well-known **pressure load**, a new surface load distribution, called '**follower load**' is introduced, distinguished from the former for its independence from the deformation of the area element on which it acts. While a uniform pressure leads to self-adjointness, this is not the case for the follower load. The propagation of waves on the loaded surface of an elastic, incompressible, and prestressed half space is analyzed for both the loads, together with the well-known case of **dead load**, also included for reference. The wave propagation problem highlights the strong difference between all these loadings, differing from each other only in the way they react to an incremental deformation. When the speed of wave propagation vanishes, surface bifurcation occurs and here a very surprising result is found. In particular, while the bifurcation takes place for compressive dead load, it occurs for a tensile pressure, and is excluded for follower load within the elliptic range. It is shown how the different loads can be exploited to obtain propagation speeds never so far achieved, thus introducing a new perspective in the design of surface acoustic wave devices.

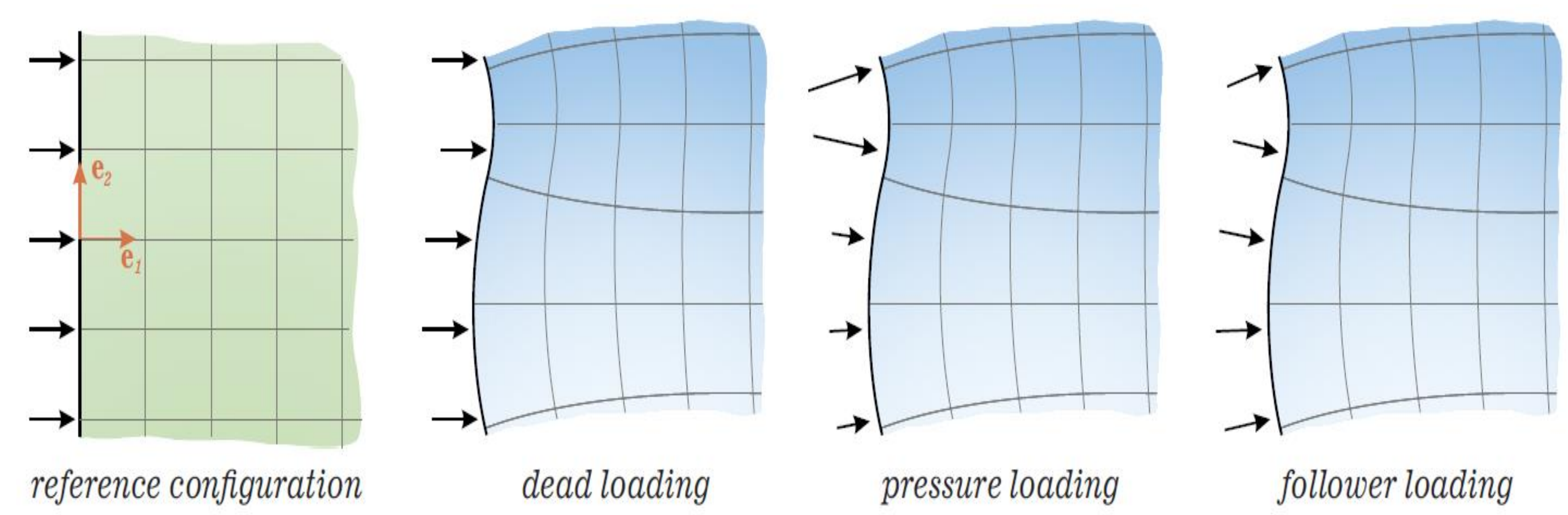

## Problem description and notation

An elastic half space $x_{01} \geq 0$ made up of a neo-Hookean material deformed in plane strain (thus neo-Hookean and Mooney-Rivlin approaches coincide) is analyzed under the effect of three loads acting orthogonally to the surface having normal unit vector $\mathbf{n}_0 = -\mathbf{e}_1$. Waves propagate along the surface and exponentially decay with the depth, so that they reduce to **Rayleigh waves** when the material is unstressed, while they describe a **quasi-static bifurcation** when their propagation speed vanishes. The case of dead loading, thoroughly analyzed (starting from Biot solution), is reported only for comparison.

Notation:

- $\mathbf{F}$ deformation gradient, $J = det\mathbf{F} = 1$ (incompressibility)
- $\mathbf{v}$ incremental displacement ($v_3 = 0$ plane strain), $v_{1,1} + v_{2,2} = 0$ (incompressibility)
- $\mathbf{n}$ normal to the surface in the current configuration $\mathbf{n}\,da = J\mathbf{F}^{-T}\mathbf{n}_0\,da_0$
- $\mathbf{T}$ Cauchy stress
- $\mathbf{t}$ nominal stress (transpose of the first Piola-Kirchhoff stress tensor) $\mathbf{t}^T\mathbf{n}_0\,da_0 = \mathbf{T}\mathbf{n}\,da$

## The three loads considered




$\mathbf{t}^T\mathbf{n}_0 = \mathbf{d}(\mathbf{x}_0)$ (dead loading), $\mathbf{t}^T\mathbf{n}_0 = -pJ\mathbf{F}^{-T}\mathbf{n}_0$ (pressure loading), $\mathbf{t}^T\mathbf{n}_0 = -f\mathbf{n}$ (follower loading)

## Incremental constitutive equations

For an incompressible material deformed under the plane strain assumption, the incremental displacement and nominal stress are related as follows (a comma denotes differentiation with respect to the spatial variables, while $\dot{\pi}$ represents the increment of the mean pressure $\pi$)

$$\begin{aligned}\dot{t}_{11} &= \mu\,(2\xi - k - \eta)\,v_{1,1} + \dot{\pi},\\ \dot{t}_{22} &= \mu\,(2\xi + k - \eta)\,v_{2,2} + \dot{\pi},\\ \dot{t}_{12} &= \mu\,[(1+k)v_{2,1} + (1-\eta)v_{1,2}],\\ \dot{t}_{21} &= \mu\,[(1-\eta)v_{2,1} + (1-k)v_{1,2}]\end{aligned}$$

$$\xi = \frac{\mu_*}{\mu}, \quad k = \frac{T_1 - T_2}{2\mu}, \quad \eta = \frac{\pi}{\mu} = \frac{T_1 + T_2}{2\mu}$$

$\mu$, $\mu_*$ incremental shear moduli (for Mooney-Rivlin material $\xi = 1$)

$$\mu > 0, \quad k^2 < 1, \quad 2\xi > 1 - \sqrt{1-k^2} > 0$$

restrictions for elliptic complex and elliptic imaginary regimes

## Incremental equilibrium equations: the stream function

The equation of motion $\dot{t}_{ij,i} + \dot{b}_j = \rho\,\ddot{v}_j$ is to be satisfied (a dot on top denotes differentiation with respect to time). Incompressibility allows the introduction of a stream function $\psi(x_1, x_2)$ with $v_1 = \psi_{,2}, \quad v_2 = -\psi_{,1}$. A representation for a wave propagating at speed $c$ in time $t$ is assumed in the separate-variables form with wave number $\kappa$

$$\psi(\mathbf{x}, t) = \psi(\mathbf{x})e^{-i\kappa\,ct}, \qquad \psi(\mathbf{x}) = a\,e^{i\kappa(h_1 x_1 + x_2)}$$

The amplitude $a$ is an arbitrary complex number, therefore the decay of the solution with depth is guaranteed when the imaginary part of $h_1$ is a positive number.

## Boundary conditions

Assuming that the loads maintain constant moduli, the increments of the nominal stress along the surface should satisfy the following conditions:

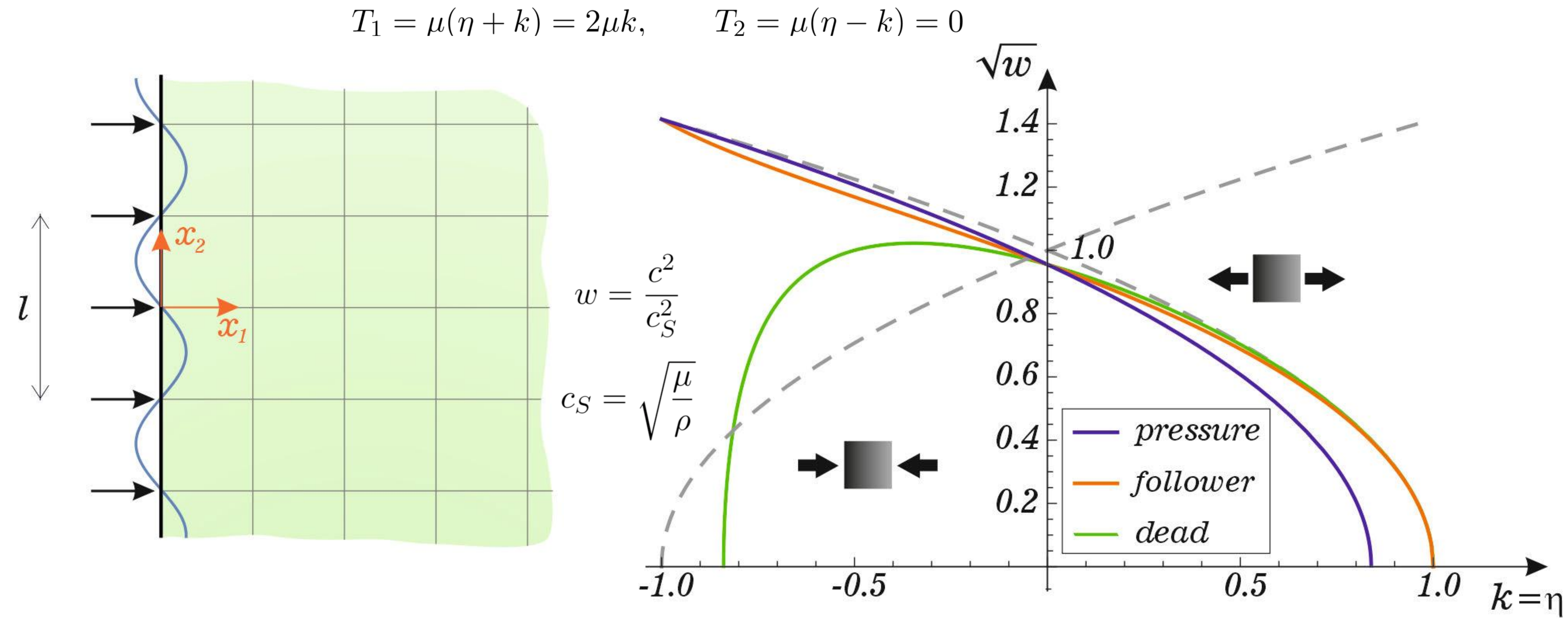
$$\dot{t}_{11} = \mu\alpha v_{1,1}, \qquad \dot{t}_{12} = \mu\beta v_{1,2},$$

| | $\alpha$ | $\beta$ |
|---|---|---|
| Dead loading | $0$ | $0$ |
| Pressure loading | $p/\mu$ | $p/\mu$ |
| Follower loading | $0$ | $f/\mu$. |

## Surface wave propagation and instability onset

The surface wave propagation is investigated within the elliptic regime ($-1 < \kappa < 1$) for each of the three loads applied as uniaxial prestress in the direction orthogonal to the loaded surface for $T_2 = 0$ ($\eta = \kappa$). Rayleigh surface waves propagate along the $\boldsymbol{e}_2$ direction (parallel to the surface), while decaying with depth for $x_1 \geq 0$. The propagation speed (represented below in dimensionless form) exhibits a peculiar evolution as a function of the intensity of the applied pre-stress; the conditions corresponding to a vanishing propagation speed, if present, indicate the onset of a surface instability.

$$T_1 = \mu(\eta + k) = 2\mu k, \qquad T_2 = \mu(\eta - k) = 0$$




The bounds for the propagation speed of volumetric waves are reported as dashed lines.

**DEAD LOAD** the surface instability shows up at compression for the pre-stress parameter $\kappa = -0.8393$ (well known Biot solution).

**PRESSURE LOAD** surprisingly, the surface instability is foreseen for the opposite condition, i.e. for a tension with $\kappa = 0.8393$! This is believed to depend on the adjustment of the traction intensity according to the variation of the area elements in the current configuration.

**FOLLOWER LOAD** the propagation speed turns out to be a decreasing function of the load intensity from compression to tension, and never vanishes within the elliptic regime.

Moreover, unlike the dead and pressure loads, the follower load is shown to lead to a non-self-adjointed problem for wave propagation, so that in principle a Hopf bifurcation or a **flutter instability** becomes possible. However, this was not found within the range of parameters investigated by the authors.


## Acknowledgments

All the authors acknowledge financial support from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (Gr. agr. N. ERC-ADG-2021-101052956-BEYOND).


## References

Please scan the qr code on the right to find an exhaustive list of references.