Bifurcation of a coated disc under different, but uniform, radial forces

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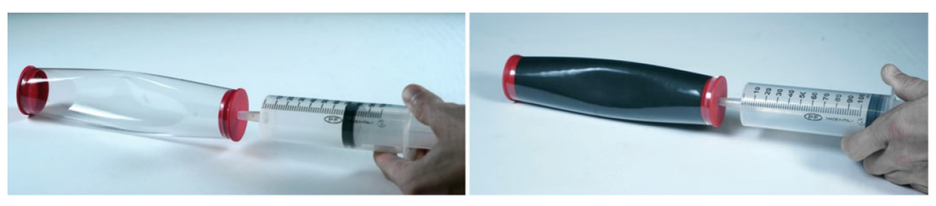
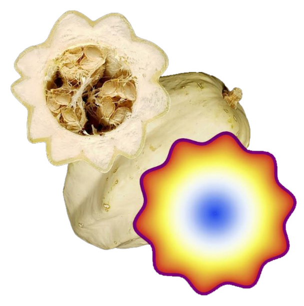
Abstract

Cylindrical structures such as rings and arches exhibit instabilities under external radial force distributions, typically evidencing an oval-shaped bifurcation mode. The presence of an internal material significantly alters this behaviour. An analytical solution is derived using a complex potential formulation [1] to describe the bifurcation behaviour of an elastic disc coated with a circular elastic rod [2]. Both perfect and incomplete bonds at the disc-coating interface are considered, and the effects of distinct types of radial loads are captured. The mechanical properties and loading conditions of the coating–disc system, determine a broad spectrum of bifurcation modes, ranging from low-order ovalization to high-order undulatory patterns. The theoretical framework and results are relevant to a wide range of applications, including coated fibre systems, and biologically inspired phenomena such as plant and fruit morphogenesis.

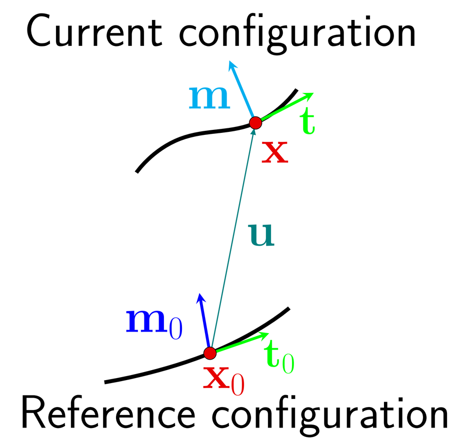


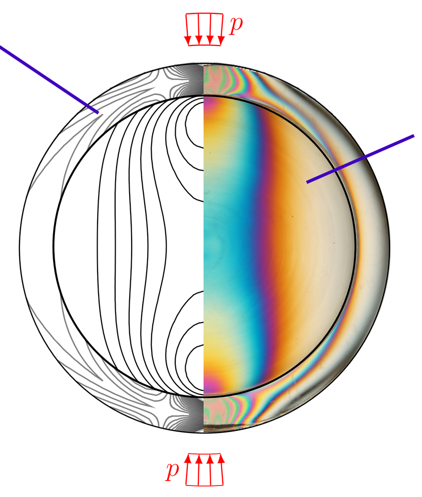


1-Mechanical model and method

When the coating is thin compared to the disc radius R, it can be modelled as a **thin interface**, a one-dimensional curve with prescribed boundary conditions, governed by the elastic rod equations. The problem is entirely solved on the coating, recognising stress transmission between the disc and the outer coating.

Interface model for the inextensible coating

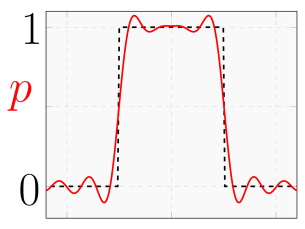




Complex variables formulation

$$\begin{cases} 2\mu u(z) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ \sigma_{11} + \sigma_{22} = 4\,\mathrm{Re}(\varphi'(z)) \\ \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2\left[\bar{z}\varphi''(z) + \psi'(z)\right] \end{cases}$$

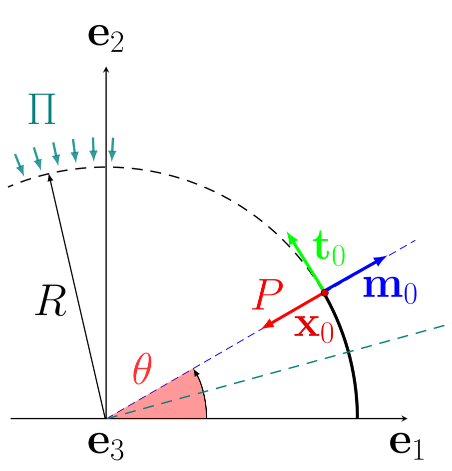
Fourier series approximation for the applied load

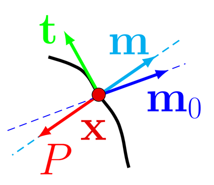


Thanks to Kolosov–Muskhelishvili complex potentials, the elastic fields on the boundary and within the disc are fully described. The **complex Fourier series method** enables the solution for any applied load distribution.

2-Radial forces applied to a annular rod

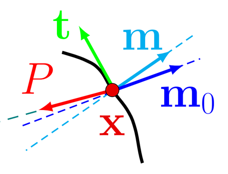
Radial and **uniform loads** leave an axially inextensible circular rod undeformed and subject to a trivial state of pure normal compressive force until buckling occurs. However, initially identical load distributions may differ in the way they react to the deformation.





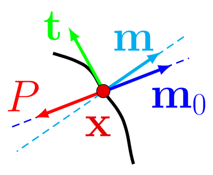
(i.) **Hydrostatic pressure**

The load is said to be hydrostatic if it is parallel to the normal unit vector $\mathbf{m}$ when passing from the reference to the current configuration.



(ii.) **Centrally-directed radial load**

The load is said to be centrally-directed if it remains always directed towards the initial centre of the ring.



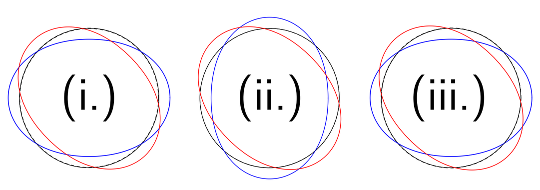
(iii.) **Dead load**

The load is said to be a "dead" radial load if it remains parallel to the normal unit vector $\mathbf{m}_0$ when passing from the reference to the current configuration.

The **critical radial load** Π_{cr}, at which bifurcation occurs in an annular rod with bending stiffness B and radius R, arises for all types of radial loading (i–iii) and under any externally applied constraint. However, the presence or absence of such a constraint modifies the value of Π_{cr}.

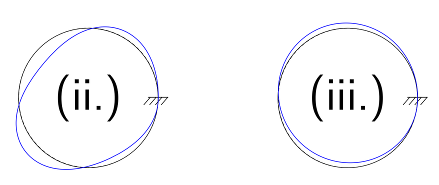
$$\Pi_{cr} = k^2 \frac{B}{R^3}$$

Critical radial load



$k^2 = 3$ $k^2 = 9/2$ $k^2 = 4$

Fully continuous solution

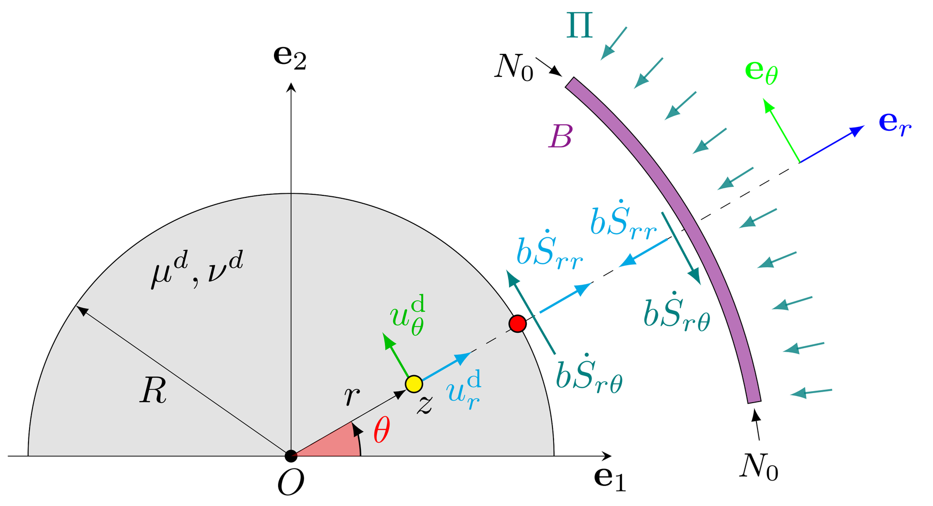


$k^2 \approx 6.472$ $k^2 = 0.701$

Effect of the external constraints

3-Incremental response for the disc/coating system

In its reference configuration and loaded with an external radial load Π, the annular rod is subject to an axial internal force $N_0 = -\Pi R$, while the interior disc remains unstressed.



At bifurcation, a non-trivial incremental deformation occurs, causing the disc to experience incremental stress and strain. The resulting incremental traction at the disc's boundary, multiplied by its thickness b, gives rise to an incremental force acting on the rod.

The incremental response of the disc/coating system is governed by the following equation

$$\frac{\partial^5 \dot{u}^c_r}{\partial\theta^5} + 2\frac{\partial^3 \dot{u}^c_r}{\partial\theta^3} + \frac{\partial \dot{u}^c_r}{\partial\theta} + \frac{\Pi R^3}{B}\left(\frac{\partial^3 \dot{u}^c_r}{\partial\theta^3} + 2\frac{\partial \dot{u}^c_r}{\partial\theta} - \dot{u}^c_\theta\right) + \mathfrak{S}^{\Pi} + \mathfrak{S}^{\sigma} = 0$$

where the terms $\mathfrak{S}^{\Pi}$ and $\mathfrak{S}^{\sigma}$ rule the type of applied radial load and the interface stress transmission properties, respectively.

4-Critical loads for the disc/coating system

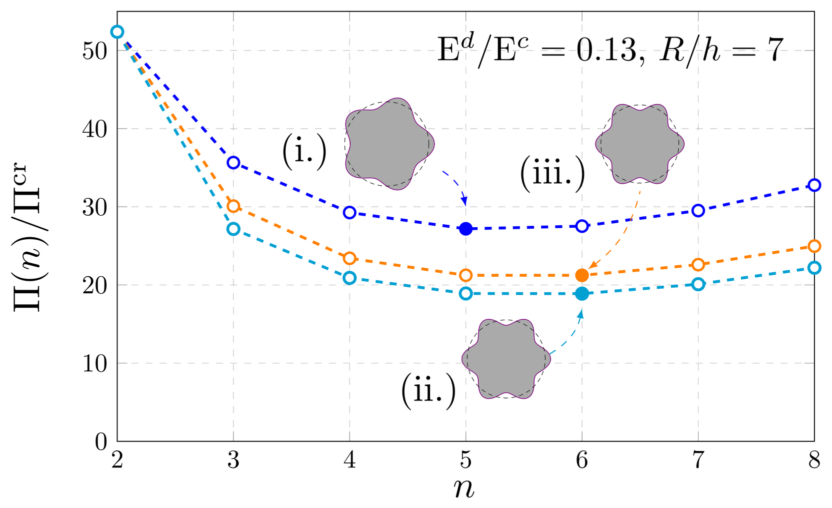
The use of a complex Fourier series expansion of the governing equations, with unknown displacement fields expressed via complex coefficients, enables the formulation of the Sturm–Liouville problem governing the buckling of the coated disk. When the complex coefficients vanish, the trivial solution is obtained, otherwise, the **bifurcation radial load for the coated disc**, corresponding to the n-th mode, is obtained

$$\frac{\Pi(n)R^3}{B} = \frac{2n^2(n^2-1) + 2\dfrac{\mu^d b R^3}{\kappa^d B}\left[(n+\mathscr{M})\kappa^d + n - \mathscr{M}\right]}{2(n^2-1) + \xi\left[(1-n)^{\alpha-1} + (1+n)^{\alpha-1}\right]}, \quad n \geq 2, \tag{1}$$

where $\mathscr{M} = 1$ ($\mathscr{M} = 0$) for perfect bonding (for slip contact) at the rod/core interface and $\xi = \alpha = 1$ for hydrostatic pressure, $\xi = 1$ and $\alpha = 0$ for centrally directed load and $\xi = \alpha = 0$ for dead radial load. Equation (1) shows that, when parameter $\mu^d b R^3/B$ tends to zero, the coated disc behaves as a rod subject to the radial load Π.

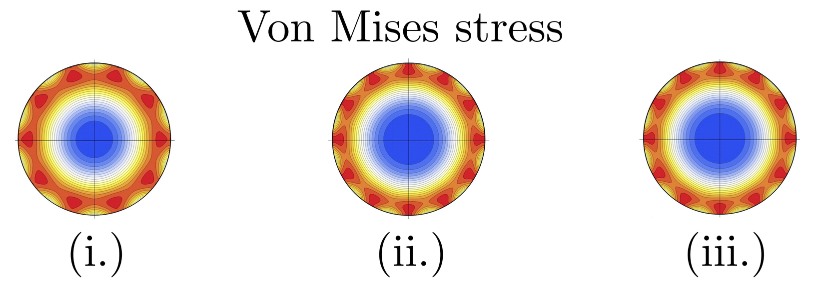
4.1-Radial forces and modes

For a given set of material and geometrical parameters and varying the mode number n in equation (1), different values for the bifurcation load can be analysed.



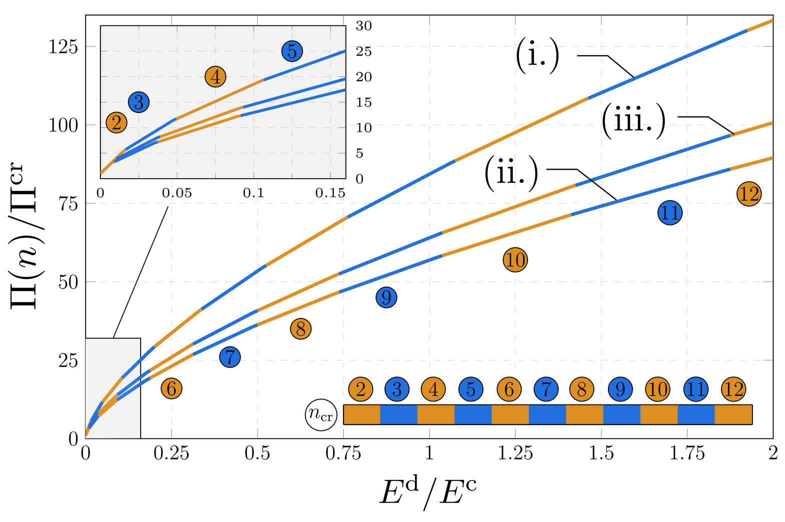
The critical value corresponds to the integer number n that minimises the equation (1). The presence of the inner disc may lead to the predominance of high-wavenumber modes. Once a value of n, corresponding to a given applied pressure $\Pi(n)$, is fixed, the Kolosov–Muskhelishvili formulae allow for the determination of all elastic fields at every point within the boundary of the disc.

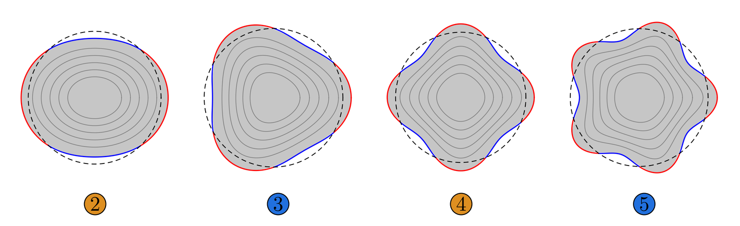
Von Mises stress



4.2-Bifurcation results

When the critical value n_{cr} is inserted inside the equation (1), the critical response for different stiffness contrasts between the disc and the coating can be investigated for the three radial load distributions and different interface conditions. The analysis reveals that the critical load increases with increasing stiffness. However, the critical loads are smaller under slip conditions than perfect bonding conditions. The increase in the critical load is accompanied by an increase in the wavenumber n of the bifurcation mode (alternance of different colour marking the curves in the following figure).





Acknowledgements

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[1] N. Muskhelishvili, Springer Science & Business Media, 1959.
[2] M. Gaibotti, S. Mogilevskaya, A. Piccolroaz, and D. Bigoni, *Proc. R. Soc. A*, vol. 480, no. 2281, p. 20230491, 2024.