New Horizons in Structural Mechanics, Elasticity and Homogenization

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Homogenization of a grid of Timoshenko beams

Matteo Franzoi[1], Davide Bigoni[1], Andrea Piccolroaz[1]

e-mail: matteo.franzoi-1@unitn.it, davide.bigoni@unitn.it, andrea.piccolroaz@unitn.it

[1] Instabilities Lab, University of Trento, Via Mesiano 77, 38123 Trento, Italy

Abstract

Two-dimensional architected materials are often implemented as periodic grids of elastic beams. Conventional homogenization methods approximate these structures as equivalent elastic solids but typically neglect shear deformation in the constituent beams. This work addresses that limitation by incorporating shear deformability via Timoshenko beam theory, enabling accurate modeling of stubby beams. Introducing shearable beams into the grid expands the design space, allowing for greater control over the effective Poisson's ratio, surpassing the limits imposed by slender beam models. Furthermore, the shear-enriched model corrects potential mispredictions of auxetic behavior that may arise when relying solely on the Euler–Bernoulli beam theory.

Introduction and notation

The classical quasi-static homogenization of two-dimensional periodic beam grids is extended by incorporating shear-deformable beam theory, enabling accurate modeling of stubby beams. The elastic properties of the resulting effective material, with out-of-plane thickness h, are expressed as functions of the beam slenderness Λ and the shear coefficient ϕ. Three approaches are possible for designing stubby beams: (i) homogenizing a discrete chain equipped with axial springs, sliders with springs and elastic hinges, resulting in a beam where all the three stiffnesses are unrelated to each other. (ii) designing a junction in which the beams lay on separate planes, but preserving the out-of-plane symmetry, allowing them to attain extremely low slendernesses. (iii) perforating a plate: this approach is limited for highly stubby beams, where the holes become so small they vanish.

$\Lambda = l\sqrt{(E_sA)/(E_sJ)} = l\sqrt{k_a/k_r}$ slenderness of the beams, $\phi = \sqrt{2(1+\nu_s)/\kappa} = \sqrt{k_a/k_t}$ shear coefficient of the beams, κ shear correction factor,

l, A, J length, cross-section area and moment of inertia of the beams, E_s, ν_s Young modulus and Poisson's ratio of the constituent material, k spring stiffness of the discrete chain.

The Timoshenko beam theory

A dash denotes differentiation with respect to the coordinate x.

The Timoshenko beam theory relaxes the Euler-Bernoulli assumption that the cross sections of a deflected beam remain perpendicular to its axis. Under the assumption of small deflections and rotations, the kinematics and the equilibrium equations, and the constitutive laws for the Timoshenko beam are

$$\varepsilon(x) = u'(x), \quad \gamma(x) = v'(x) - \theta(x), \quad \chi(x) = \theta'(x)$$
$$N' = 0, \quad V' = 0, \quad M' = -V,$$
$$N = E_sA\,\varepsilon, \quad V = \kappa G_sA\,\gamma, \quad M = E_sJ\,\chi.$$

The elastic strain-energy of the Timoshenko beam is

$$\mathcal{E} = \int_0^l \left(E_sA\,\varepsilon^2 + \kappa G_sA\,\gamma^2 + E_sJ\,\chi^2\right) dx.$$

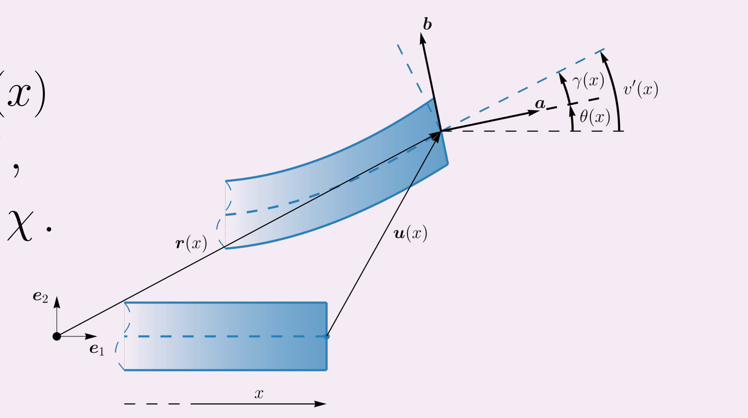

Fig. 1: Deformed configuration of the planar Timoshenko beam at coordinate x.

Homogenized discrete chain

The chain is created by periodically repeating a unit cell of length a.

The homogenization of a discrete periodic chain of rigid elements, equipped with elastic hinges of stiffness k_r, axial springs of stiffness k_a and sliders with spring of stiffness k_t, has been shown to yield a continuous beam model, including the Euler-Bernoulli and the Timoshenko beam models.

$$\varepsilon_i = \frac{u_i - u_{i-1}}{a}, \quad \gamma_i = \frac{v_i - v_{i-1}}{a} - \theta_i,$$
$$i = 1, \dots, n,$$
$$\chi_i = \frac{\theta_{i+1} - \theta_i}{a}, \quad i = 1, \dots, n-1,$$
$$\mathcal{E}^d = \frac{a^2}{2}\sum_{i=1}^{n}\left(k_a\varepsilon_i^2 + k_t\gamma_i^2\right) + \frac{a^2}{2}\sum_{i=1}^{n-1} k_r\chi_i^2$$

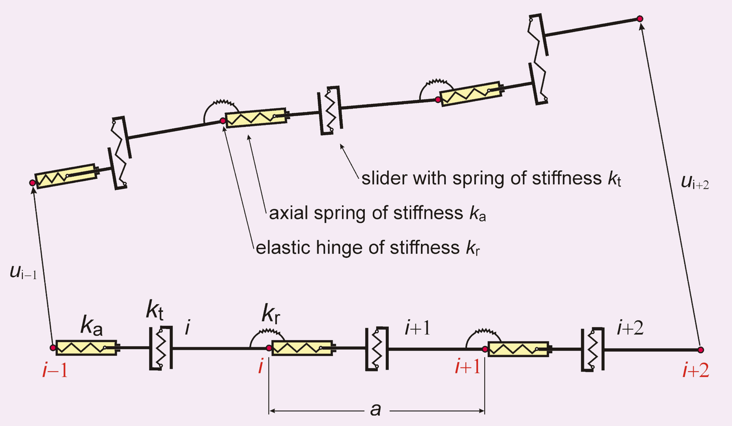

Fig. 2: A microstructured chain model.

In the limit where $a \to 0$ and $n \to \infty$, $\mathcal{E}^d \to \mathcal{E}$ so that $k_a\,a \to E_sA$, $k_t\,a \to \kappa G_sA$, $k_r\,a \to E_sJ$.

Quasi-static homogenization procedure

For the square lattice sketched in Fig. 3, $\tilde{\boldsymbol{q}} = [\tilde{\boldsymbol{q}}^{(lb)}, \tilde{\boldsymbol{q}}^{(rb)}, \tilde{\boldsymbol{q}}^{(lt)}]^\mathsf{T}$, $\tilde{\boldsymbol{q}}^* = \tilde{\boldsymbol{q}}^{(lb)}$, and $\boldsymbol{Z}_0 = [\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}]^\mathsf{T}$

A periodic beam grid can be created by tessellating a unit cell along the directions defined by the direct basis vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. The quasi-static homogenization procedure is based on the matching between the elastic strain-energy density of the unit cell and the one of a Cauchy elastic continuum.

$$\frac{1}{|\mathcal{C}|}\boldsymbol{q}(\tilde{\boldsymbol{q}}(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}) \cdot \boldsymbol{K}\,\boldsymbol{q}(\tilde{\boldsymbol{q}}(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} \cdot \mathbb{E}[\boldsymbol{\varepsilon}] \quad \Longrightarrow \quad \mathbb{E} = \frac{1}{2|\mathcal{C}|}\frac{\partial^2\Big(\boldsymbol{q}(\tilde{\boldsymbol{q}}^*(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon}) \cdot \boldsymbol{K}\,\boldsymbol{q}(\tilde{\boldsymbol{q}}^*(\boldsymbol{\varepsilon}),\boldsymbol{\varepsilon})\Big)}{\partial\boldsymbol{\varepsilon}^2}$$

where $\mathcal{C}$ is the area of the unit cell, $\boldsymbol{K}$ is the stiffness matrix of the unit cell, $\boldsymbol{q} = \tilde{\boldsymbol{q}} + \hat{\boldsymbol{q}}(\boldsymbol{\varepsilon})$ is the generalized displacement vector — decomposed into a periodic field $\tilde{\boldsymbol{q}} = \boldsymbol{Z}_0\tilde{\boldsymbol{q}}^*$ and an affine deformation such that, for any node j of the unit cell, $\hat{\boldsymbol{q}}^{(j)} = \boldsymbol{\varepsilon}\boldsymbol{x}^{(j)}$ — and $\mathbb{E}$ is the fourth-order elasticity tensor.

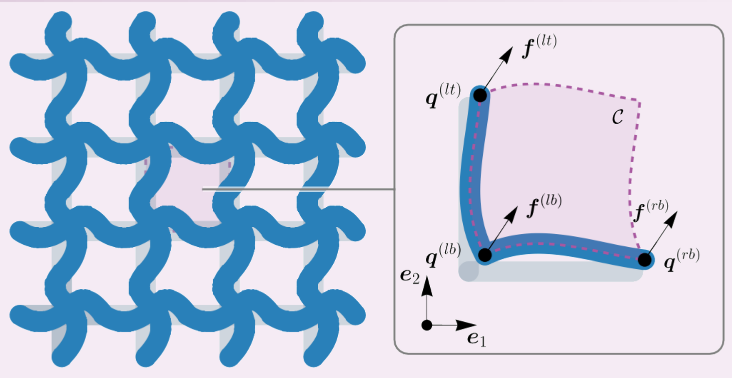

Fig. 3: Deformed configuration for a square lattice and its unit cell, chosen to minimize the number of beams.

Material equivalent to a hexagonal lattice

For an isotropic constituent material, the Euler-Bernoulli model coincides with the Timoshenko model when $\nu_s = -1$.

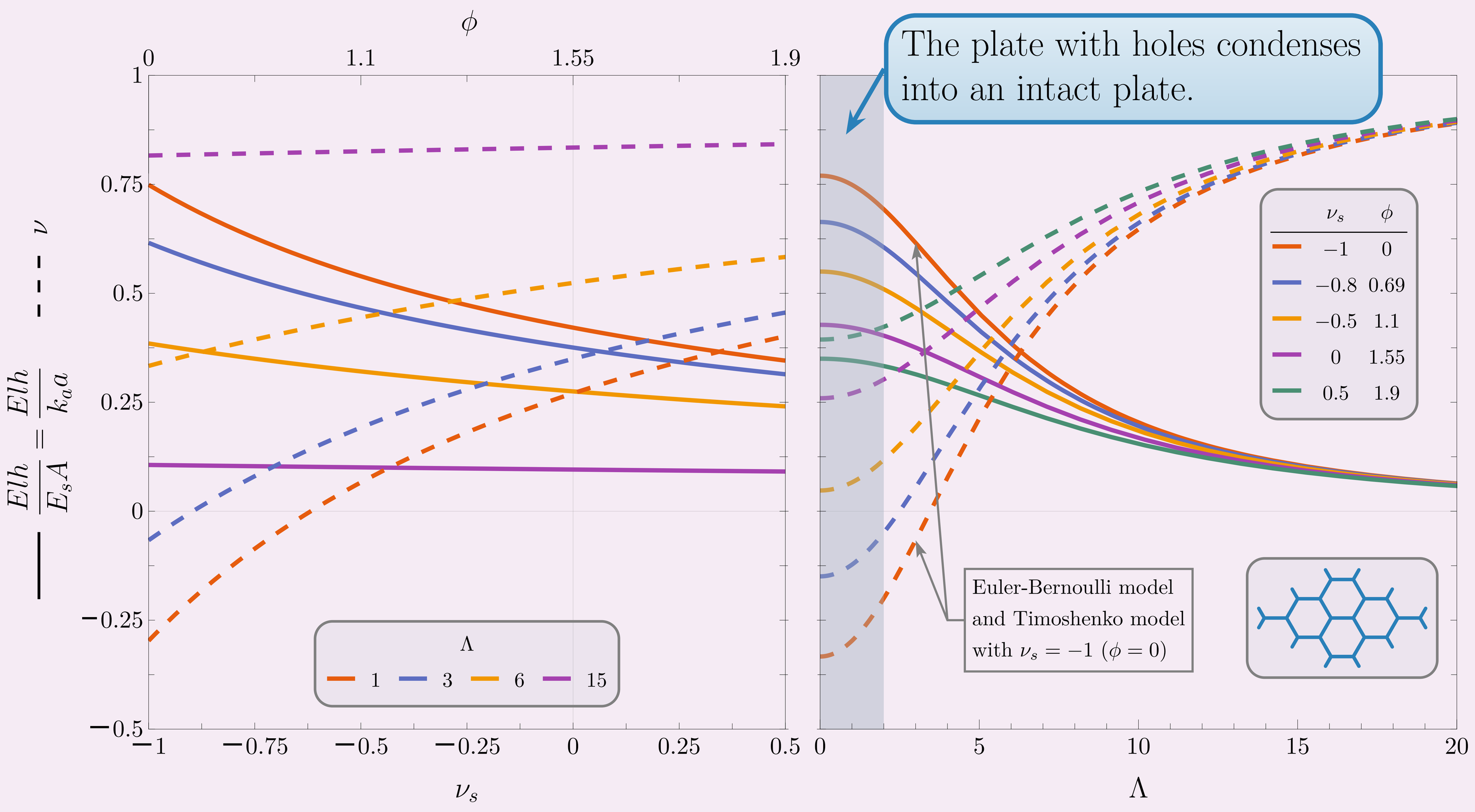

Fig. 4: The dimensionless Young modulus and the Poisson's ratio of the material equivalent to an hexagonal grid of Timoshenko beams are plotted as functions of the Poisson's ratio of the constituent material, ν_s (or equivalently, ϕ), and the slenderness, Λ, for $\kappa = 5/6$.

$$\frac{Elh}{E_sA} = \frac{16\sqrt{3}}{\Lambda^2 + 12(\phi^2+3)}, \qquad \nu = \frac{\Lambda^2 + 12(\phi^2-1)}{\Lambda^2 + 12(\phi^2+3)}$$

The elastic constants of the equivalent material can be tuned by properly selecting Λ and ϕ (or ν_s). The gray zone becomes attainable when the microstructured beam in Fig. 2 or the junction in Fig. 5 is used.

Conceptual model of the node for stubby beams

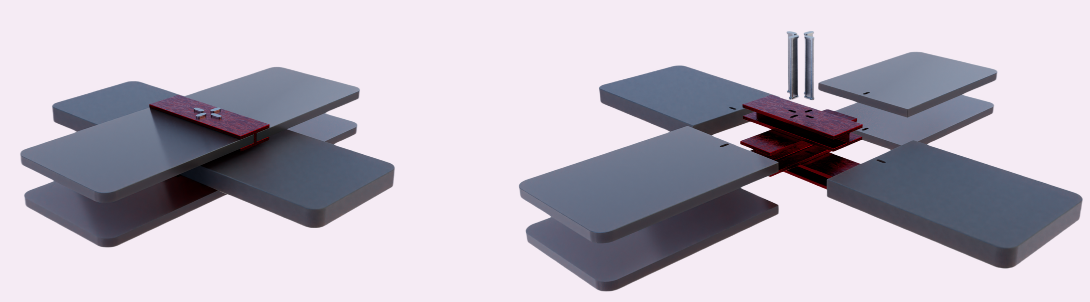

Fig. 5: Proof-of-concept design of a junction of 4 stubby beams, operating in separate planes without interference, but preserving out-of-plane symmetry. Perspective view (left) and exploded view (right). The junction imposes full continuity of displacement between the connected beams. Each beam is connected to the junction through orthogonal 'pivots' that prevent relative rotation while allowing the Poisson's effect.

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References

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