Dynamics of origami-like microstructured elastic rods

S.P.C. Dhanakoti[1], M. Paradiso[2], D. Bigoni[1], F. Dal Corso[1]
[1]Dipartimento di Ingegneria Civile Ambientale e Meccanica, University of Trento, Italy.
[2]Nanyang Technological University, Singapore, SG.

Introduction

The nonlinear dynamics of a shearable elastic rod, modelled through homogenisation of microstructure composed of elastic hinges and four-bar linkages, is studied. This microstructure enables the shear deformation and is capable of producing folding and faulting patterns [1]. Moreover, this microstructure allows only length shortening. In the present contribution, we extend this work to investigate the dynamical aspects.

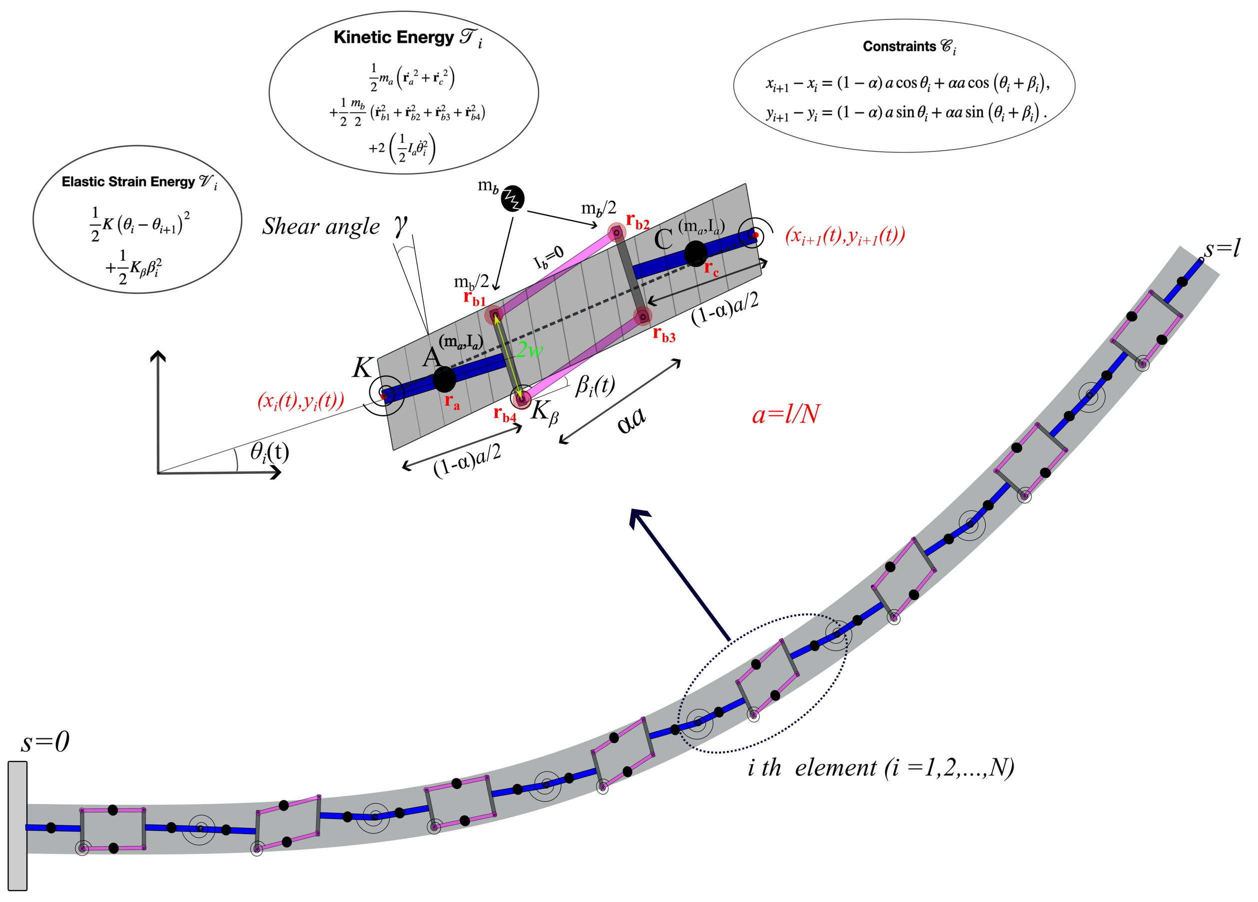


Figure: Microstructured one-dimensional structure and equivalent shearable elastic rod.

- $\dot{()} = \frac{\partial}{\partial t}$ and $()' = \frac{\partial}{\partial s}$, where $s \in [0, l]$ is arclength parameter and $t \in [0, \infty]$ is time.

Equilibrium Equations

Discrete Model

Minimize $\mathscr{L} = \sum_{i=1}^{N} \mathscr{T}_i - \mathscr{V}_i$

Subject to $\cup_{i=1}^{N} \mathscr{C}_i$

Lagrange Multipliers N_x^i, N_y^i

$$-K\left(2\theta_i - \theta_{i-1} - \theta_{i+1}\right) - N_x^i\left((1-\alpha)a\sin\theta_i + \alpha a \sin\left(\theta_i + \beta_i\right)\right)$$
$$-N_y^i\left(-(1-\alpha)a\cos\theta_i - \alpha a\cos\left(\theta_i+\beta_i\right)\right) - \left(2m_b w^2 + (\ldots)a^2\right)\ddot{\theta}_i + a\left(..\dot{x}_i + ..\dot{y}_i\right)\dot{\theta}_i = 0,$$
$$-K_{sh}\beta_i - N_x^i \alpha a \sin\left(\theta_i+\beta_i\right) + N_y^i \alpha a\cos\left(\theta_i + \beta_i\right) = 0$$
$$N_x^{i+1} - N_x^i - 2m_a\ddot{x}_i - \frac{1}{2}m_b\ddot{x}_{i-1} - m_b\ddot{x}_i - \frac{1}{2}m_b\ddot{x}_{j+1} + (\ldots)a\ddot{\theta}_i + (\ldots)a\dot{\theta}_i^2 + (\ldots)a\ddot{\theta}_{i+1} + (\ldots)a\dot{\theta}_{i+1}^2 = 0,$$
$$N_y^{i+1} - N_y^i - 2m_a\ddot{y}_i - \frac{1}{2}m_b\ddot{y}_{i-1} - m_b\ddot{y}_i - \frac{1}{2}m_b\ddot{y}_{j+1} + (\ldots)a\ddot{\theta}_i + (\ldots)a\dot{\theta}_i^2 + (\ldots)a\ddot{\theta}_{i+1} + (\ldots)a\dot{\theta}_{i+1}^2 = 0,$$

$N \to \infty$

Continuum Model

$$-\left(\rho_b w^2 + \rho_a J\right)\ddot{\theta} + EI\theta'' - N_x\left((1-\alpha)\sin\theta + \alpha\sin\left(\theta + \gamma/\alpha\right)\right) + N_y\left((1-\alpha)\cos\theta + \alpha\cos\left(\theta+\gamma/\alpha\right)\right) = 0,$$
$$-GA_s\gamma - N_x\sin\left(\theta+\gamma/\alpha\right) + N_y\cos\left(\theta+\gamma/\alpha\right) = 0,$$
$$N_x' - \rho\ddot{x} = 0, \qquad N_y' - \rho\ddot{y} = 0,$$
$$x' - (1-\alpha)\cos\theta - \alpha\cos(\theta+\gamma/\alpha) = 0,$$
$$y' - (1-\alpha)\sin\theta - \alpha\sin(\theta+\gamma/\alpha) = 0.$$

- The continuum quantities ρ, ρ_b and γ are the linear mass densities and shear angle, respectively

$$\rho = \lim_{a\to 0}\frac{2(m_a+m_b)}{a}, \qquad \rho_b = \lim_{a\to 0}\frac{2m_b}{a}, \qquad \rho_a J = \lim_{a\to 0}\frac{2I_a}{a}, \qquad \gamma = \alpha\lim_{a\to 0}\beta^i.$$

- The stiffnesses K, and K_β in the discrete model are related to the material stiffnesses EI and GA_s of the continuum through

$$K = \frac{EI}{a}, \qquad K_\beta = a\alpha^2 GA_s.$$

Dynamics: Linear Analysis

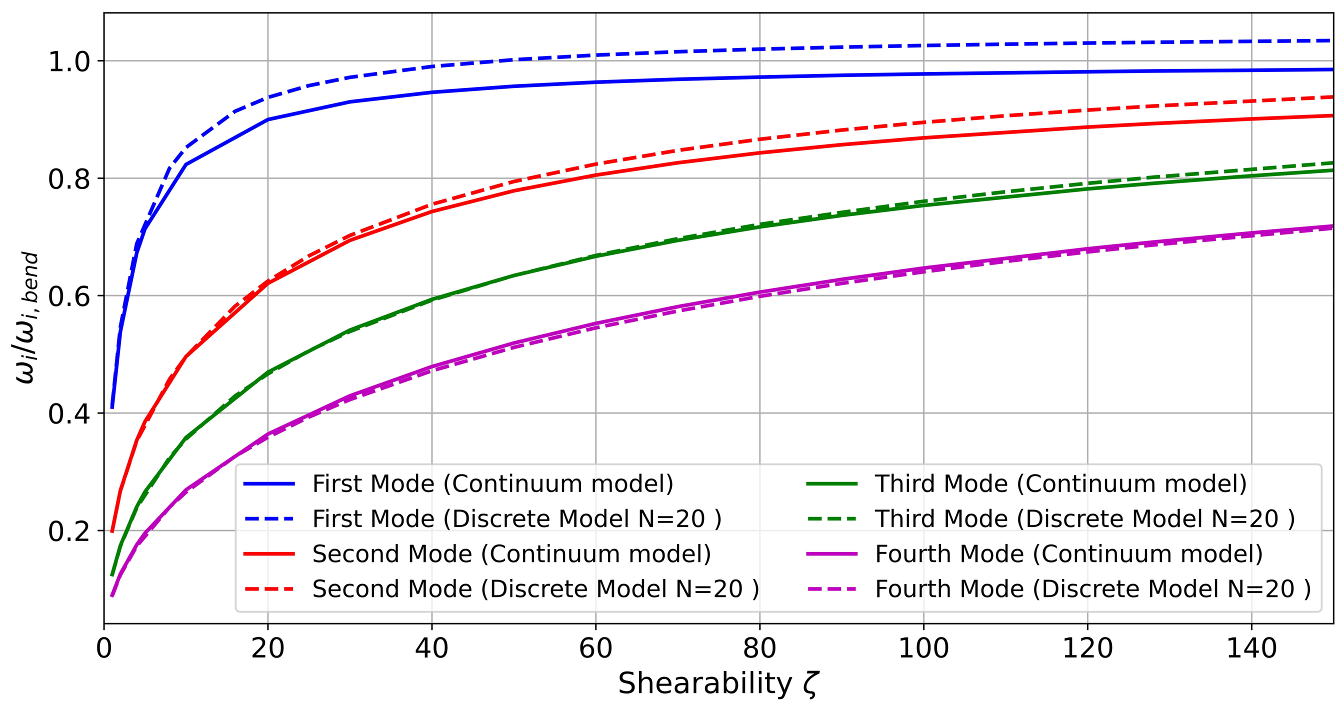
- For small deformations, $x \approx s$ and $y' \approx \theta + \gamma$.
- On non-dimensionalization by defining $\bar{x} = x/l$, $\bar{y} = y/l$, $\tau = \sqrt{\frac{EI}{\rho l^2}}t$ and $\zeta = \frac{GA_s l^2}{EI}$, we obtain

$$\frac{\partial^4 y}{\partial \bar{x}^4} - \left(\frac{\rho_b w^2 + \rho_a J}{\rho l^2} + \frac{1}{\zeta}\right)\frac{\partial^4 y}{\partial t^2 \partial x^2} + \frac{\left(\rho_b w^2 + \rho_a J\right)}{\rho l^2}\frac{1}{\zeta}\frac{\partial^4 \bar{y}}{\partial \tau^4} + \frac{\partial^2 \bar{y}}{\partial \tau^2} = 0$$

These equations are same as that of Timoshenko beam and Rayleigh beam.

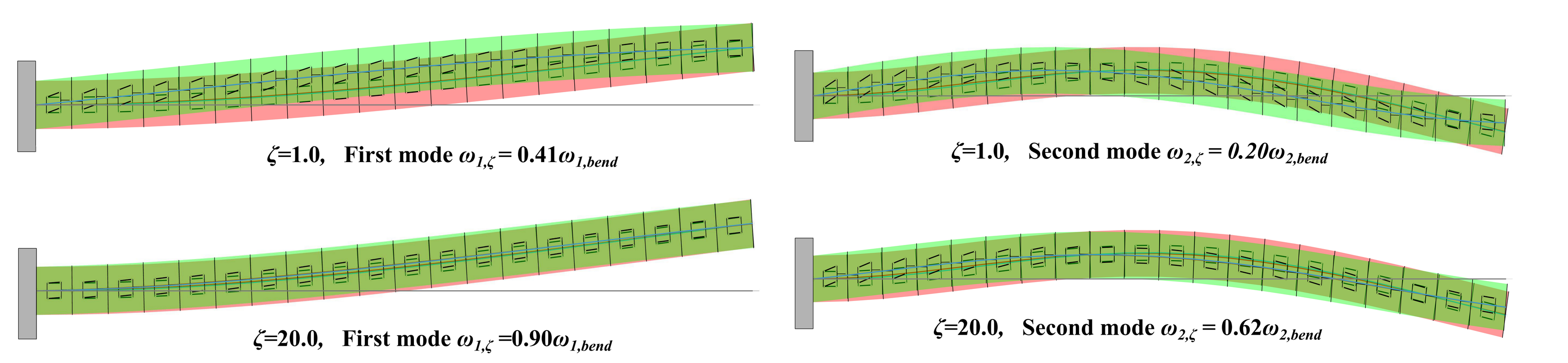
Vibration Modes of a rod: Clamped-Free ends

- The effect of ζ on the modes of vibrations



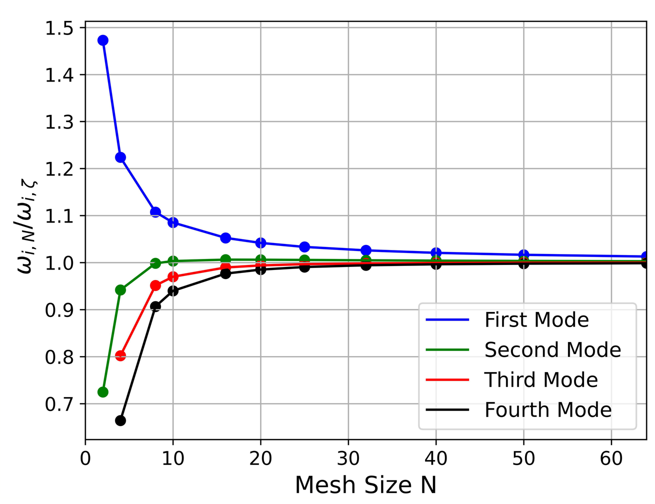
where $\omega_{i,bend}$ is the natural frequency of i th mode for a pure bending case. The rotational inertial Γ defined as $\frac{\rho_a J + \rho_b w^2}{\rho l^2}$ is taken as zero.

- The effect of ζ on the mode shapes

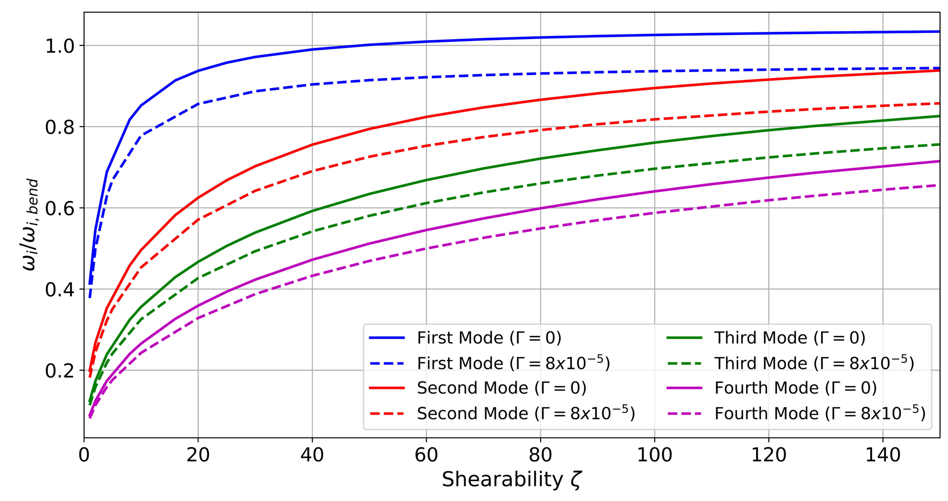


- The vibrating modes for various ζ are shown in green, while vibrating modes of pure bending are shown in red for comparision.
- The vibrating modes corresponding to discrete model for N=20 are displayed through overlapping four-bar linkages.

- The effect of mesh size N and the rotational inertia Γ



Fixed ζ=20, Γ=0



Fixed N=20

where $\omega_{i,\zeta}$ is the natural frequency of i th mode of continuum model for a given ζ and $\omega_{i,N}$ is the natural frequency of i th mode for a N sized discrete model.

Conclusion and Future Studies

- The continuum model obtained through homogenization of the origami microstructure effectively represents the dynamics under small deformations.
- Nonlinear dynamical analysis and forced vibrations

Acknowledgments

SPCD and FDC acknowledge funding from the Italian Ministry of University and Research in the framework of the Call for Proposals for scrolling of final rankings of the PRIN 2022 call - Protocol no. 2022WFJ795. MP and DB acknowledge funding from the European Research Council (ERC) under the European Union's Horizon Europe research and innovation programme, Grant agreement No. ERC-ADG-2021-101052956-BEYOND

Reference

[1] Paradiso, M., Dal Corso, F., Bigoni, D. (2025). A nonlinear model of shearable elastic rod from an origami-like microstructure displaying folding and faulting. JMPS.